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SOME GEOMETRIC INEQUALITIES OF MATHEMATICAL CONDUCTANCE

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Abstract. Let $D_0, D_1 \subset \overline{R}^n$ be non-empty sets and let Γ be the family of all closed curves which join D_0 to D_1. In this note, we introduce the concept of the mathematical conductance $C(\Gamma)$ of a curve family Γ and examine some basic properties of mathematical conductance. And we obtain the inequalities in connection with capacity of condensers.

1. Introduction

The mathematical conductance of a curve family is a basic tool in the theory of conformal mappings. The numerical value of the mathematical conductance is known only for a few curve families. Therefore good estimates are of importance. Several estimates are given in the paper ([1], [5], [6], [9]). And in Gehring [3], he has shown that the capacity is related to the mathematical conductance of a family of surfaces that separate the boundary components of a space ring E.

Throughout this paper, n is a fixed integer and $n \geq 2$. We denote the n-dimensional Euclidean space by R^n and its one-point compactification by $\overline{R}^n = R^n \cup \{\infty\}$. All topological operations are performed with respect to $\overline{R}^n$. Balls and spheres centered at $x \in R^n$ and with radius $r > 0$ are denoted, respectively, by

$$B^n(x, r) = \{y \in R^n : |y - x| < r\}$$

$$S^{n-1}(x, r) = \partial B^n(x, r) = \{y \in R^n : |y - x| = r\}$$

We employ the abbreviations

$$B^n(r) = B^n(0, r), \quad B^n = B^n(1),$$

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$$S^{n-1}(r) = S^{n-1}(0, r), \quad S^{n-1} = S^{n-1}(1).$$

As a measure in R^n we use the n-dimensional m_n, where the subscript n may be omitted. And we abbreviate $\omega_n = m_n(B^n)$, where

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{G(1+\frac{n}{2})}, \ (G : gamma\ function).$$

2. Mathematical conductance

Definition 2.1. Given a family, Γ, of nonconstant curves γ in $\overline{R}^n$, we let $bmf(\Gamma)$ denote the family of Borel measurable functions $\rho : R^n \to [0, \infty)$ such that

$$\int_\gamma \rho \ ds \tag{2.1}$$

for all locally rectifiable $\gamma \in \Gamma$. We call

$$C(\Gamma) = inf_{\rho \in bmf(\Gamma)} \int_{R^n} \rho^n \ dm \tag{2.2}$$

the mathematical conductance of Γ.

Example 2.2 ([11]). Let T be the rectangular parallelepiped with two parallel faces P_1,P_2. If Γ is the family of curves γ joining two parallel faces P_1 and P_2 of area A with distance d, then

$$C(\Gamma) = A \cdot d^{1-n}. \tag{2.3}$$

In fact, choose a Borel measurable functions $\rho \in bmf(\Gamma)$ and let γ_y be the vertical segment which join P_1 and a point y in the base P_2. Then $\gamma_y \in \Gamma$ and

$$1 \le \left(\int_\gamma \rho \ ds \right)^n \le d^{n-1} \int_{\gamma_y} \rho^n \ ds.$$

This holds for all such y and hence

$$\int_T \rho^n \ dm \ge \int_{P_2} \left(\int_{\gamma_y} \rho^n \ ds \right) dm_{n-1} \ge A \cdot d^{1-n}.$$

Since ρ is arbitrary,

$$C(\Gamma) \ge A \cdot d^{1-n}.$$

Next, let

$$\rho = \frac{1}{d}$$

be inside the parallelepiped T and $\rho = 0$ otherwise.

Then $\rho \in bmf(\Gamma)$ and

$$C(\Gamma) \leq \int_T \rho^n \; dm = A \cdot d^{1-n}.$$

EXAMPLE 2.3. If Γ is the family of curves joining the sphere with center x_0 and radius r_1 to the concentric sphere of radius r_2, then

$$C(\Gamma) = n\omega_n \left(log\frac{r_2}{r_1} \right)^{1-n} . \tag{2.4}$$

Proof. Choose $\rho \in bmf(\Gamma)$ and let

$$\gamma_e = \{x | x = re, \; r_1 < r < r_2\}$$

be the radial segment in Γ and parallel to the unit vector e. Using Hölder's inequality (See [4], theorem 189, P.140) we obtain

$$1 \leq \left(\int_{\gamma_e} \rho \; ds \right)^n \leq \left(log\frac{r_2}{r_1} \right)^{n-1} \int_{r_1}^{r_2} \rho^n \; r^{n-1} \; dr.$$

Integrating over all e we obtain by Fubini's theorem in polar coordinates

$$n\omega_n \leq \left(log\frac{r_2}{r_1} \right)^{n-1} \int_{E^*} \rho^n \; dm,$$

where E^* is the spherical ring $r_1 < |x| < r_2$. The equality holds for

$$\rho = \frac{1}{|x| log\frac{r_2}{r_1}}.$$

Thus

$$C(\Gamma) = n\omega_n \left(log\frac{r_2}{r_1} \right)^{1-n} .$$

□

PROPOSITION 2.4 ([10]). *If each curve γ_1 in a family Γ_1 contains a subcurve γ_2 in a family Γ_2, then*

$$C(\Gamma_1) \leq C(\Gamma_2).$$

In fact, choose a Borel measurable functions $\rho \in bmf(\Gamma_2)$ and suppose $\gamma_1 \in \Gamma_1$ is locally rectifiable. Then

$$\int_{\gamma_1} \rho \; ds \geq \int_{\gamma_2} \rho \; ds,$$

where γ_2 is the subcurve in Γ_2, and $\rho \in bmf(\Gamma_1)$. Thus

$$C(\Gamma_1) \leq \int_{R^n} \rho^n \; dm$$

and taking the infimum over all such ρ yields

$$C(\Gamma_1) \le C(\Gamma_2). \tag{2.5}$$

Consequently, the set of fewer and longer curves has the smaller mathematical conductance.

PROPOSITION 2.5. *For curve family* Γ_j,

$$C\left(\cup_j \Gamma_j\right) \le \sum_j C(\Gamma_j).$$

Proof. We may assume $C(\Gamma_j) < \infty$ for all j. Then given $\varepsilon > 0$ we can choose a $\rho_j \in bmf(\Gamma_j)$ such that

$$\int_{R^n} (\rho_j)^n \, dm \le C(\Gamma_j) + 2^{-j}\varepsilon.$$

Now let

$$\rho = \sup_j \rho_j, \qquad \Gamma = \cup_j \Gamma_j.$$

Then $\rho : R^n \to [0, \infty)$ is Borel measurable. Moreover, if $\gamma \in \Gamma$ is locally rectifiable, then $\gamma \in \Gamma_j$ for some j,

$$\int_\gamma \rho \, ds \ge \int_\gamma \rho_j \, ds \ge 1$$

and hence $\rho \in bmf(\Gamma)$ by definition 2.1. Thus

$$\begin{aligned} C(\cup_j \Gamma_j) &= C(\Gamma) \\ &\le \int_{R^n} \rho^n \, dm \le \int_{R^n} \sum_j (\rho_j)^n \, dm \le \sum_j C(\Gamma_j) + \varepsilon. \end{aligned} \tag{2.6}$$

□

PROPOSITION 2.6 ([1]). *If* $f : \overline{R}^n \to \overline{R}^n$ *is a one to one conformal mapping, then*

$$C(f(\Gamma)) = C(\Gamma). \tag{2.7}$$

for all curve families Γ *in* $\overline{R}^n$.

In fact, choose a Borel measurable function $\rho' \in bmf(f(\Gamma))$, let

$$\rho(x) = \rho' \circ f(x)|f'(x)|$$

for $x \in R^n - \{f^{-1}(\infty)\}$, and let Γ_0 be the family of $\gamma \in \Gamma$ which pass through $f^{-1}(\infty)$. Then

$$C(\Gamma) = C(\Gamma - \Gamma_0), \quad \rho \in bmf(\Gamma - \Gamma_0)$$

and hence

$$C(\Gamma) \le \int_{R^n} \rho^n \; dm = \int_{R^n} (\rho' \circ f)^n |f'| \; dm$$
$$= \int_{R^n} (\rho' \circ f)^n J(f) \; dm$$
$$= \int_{R^n} (\rho')^n \; dm.$$

Taking the infimum over every such ρ' gives

$$C(\Gamma) \le C(f(\Gamma)).$$

The opposite inequality follows by repeating the preceding argument with f replaced by f^{-1}.

3. Capacity of condensers

A condenser is a ring $E \subset \overline{R}^n$ whose complement is the union of two distinguished disjoint compact sets D_0 and D_1 in $\overline{R}^n$. We write

$$E = E(D_0, D_1).$$

Thus, ring is a condenser $E = E(D_0, D_1)$ where D_0 and D_1 are continua. We call D_0 and D_1 the complementary components of E.

Definition 3.1 ([9]). We let $d(x, y)$ denote the chordal distance between points $x, y \in \overline{R}^n$. That is

$$d(x, y) = |x - y| \cdot [(1 + |x|^2)(1 + |y|^2)]^{-\frac{1}{2}}, \quad x, y \neq \infty$$

Let $bmf(E)(\neq \emptyset)$ denote the family of functions $u : \overline{R}^n \to R^1$ with the following conditions :

(i) u is continuous in $\overline{R}^n$ and u has distribution derivatives in R^1,
(ii) $u = 0$ on D_0, $u = 1$ on D_1,
(iii) $u(x) = \min\{\frac{d(x,D_0)}{d(D_1,D_0)},\ 1\} \in bmf(E)$.

We call

$$Cap(E) = \inf_{u \in bmf(E)} \int_E |\nabla u|^n \; dm \tag{3.1}$$

the capacity of E.

Theorem 3.2. *If $E = E(D_0, D_1)$ is a condenser and if Γ is the family of curves γ joining D_0 and D_1 in E, then*

$$Cap(E) \leq C(\Gamma). \tag{3.2}$$

Proof. Choose a bounded continuous Borel measurable function $\rho \in bmf(\Gamma)$ and let

$$u(x) = \min\{1, \inf_{\gamma} \int_{\gamma} \rho \ ds\}$$

for $x \in E$, where the infimum is taken over all locally rectifiable γ joining D_0 to x in E. Then u has distribution derivatives and

$$\lim_{x \to D_0} u(x) = 0, \qquad \lim_{x \to D_1} u(x) = 1.$$

Hence we can extend u to $\overline{R}^n$ so that $u \in bmf(E)$. Then since $|\bigtriangledown u| = \rho$ in E,

$$Cap(E) \leq \int_E \rho^n \ dm \leq \int_{R^n} \rho^n \ dm.$$

Another smoothing argument shows the infimum over such ρ gives $C(\Gamma)$. Thus

$$Cap(E) \leq C(\Gamma).$$

□

References

[1] P. Caraman, *n-Dimensional Quasiconformal Mappings* J. Editura Academic Bucuresti, Romania, 1974.

[2] E. Villamor, *Geometric proofs of some classical results on boundary values for analytic functions*, J. Canadian Mathematical Bulletin. **37** (1994), 263–269.

[3] F. W. Gehring, *Quasiconformal mappings*, J. Complex analysis and its applications. **11** (1976), 213–268.

[4] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Univ. Press, Cambridge, 1988.

[5] O. Martio, S. Rickman, and J. Vaisala, *Definitions for quasiregular mappings*, J. Ann. Acad. Sci. Fenn. Ser. **448** (1969), 1–40.

[6] R. Nakki, *Extension of Loewner's capacity theorem*, J. Trans. Amer. Math. Soc. **180** (1973), 229–236.

[7] M. D. O'neill and R. E. Thurman, *Extremal problems for Robin capacity*, J. Complex Variables Theory and Applications **41** (2000).

[8] Y. Shen, *Extremal problems for quasiconformal mappings*, J. Mathematical Analysis and Applications. **247** (2000), 27–44.

[9] J. Vaisala, *Lectures on n-Dimensional Quasiconformal Mappings*, J. Springer-Verlag, New York, 1971.

[10] B. H. Chung, *Some applications of extremal length to conformal imbeddings*, J. Chungcheong. Math. Soc. **22** (2009), 211–216.

[11] B. H. Chung, *Extremal length and geometric inequalities*, J. Chungcheong Math. Soc. **20** (2007), 147–156.

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